JOURNAL OF APPROXIMATION THEORY 32, 226–232 (1981)

A Natural Simple Projection with Norm $\leqslant \sqrt{n}$

BRUCE L. CHALMERS

Department of Mathematics, University of California, Riverside, California 92521

Communicated by E. W. Cheney

Received September 28, 1979

Let V be an n-dimensional subspace of a Banach space X. There is a natural, easily constructed projection from X onto V with norm $\leqslant \sqrt{n}$.

0. INTRODUCTION AND PRELIMINARIES

Kadec and Snobar, in an elegant paper [5], have demonstrated the existence of a projection with norm $\leqslant \sqrt{n}$ from an arbitrary Banach space X to an n-dimensional subspace V. ($\sqrt{n}$ is in general best possible.) Their proof, however, depends on an important technical lemma due to John [4] and is nonconstructive.

In this paper we exhibit a natural, easily constructed projection $P_{X,V}$ with norm $\leqslant\sqrt{n}$. A bounded projection P can be identified with an n-dimensional subspace $P = [g_1,..., g_n] = [g] \subset X^*$, i.e., $Px = \langle x, g\rangle \cdot v = \sum_{i=1}^{n} \langle x, g_i\rangle v_i$, where $V = [v_1,..., v_n] = [v]$ and v is dual to g (i.e., $\langle v_i, g_j\rangle = \delta_{ij}$). P will also be represented by its "kernel" $g \cdot v = \sum_{i=1}^{n} g_i v_i$ when the dual basis v is required explicitly.

We can give a simple formula (see (1) below) for $P_{X,V}$ if first we make the usual isomorphic identification of X with a subspace of $C(Q)$, the space of continuous functions on $Q = ESX^*$, the *E*xtreme points of the unit *S*phere in X^*, the dual of X. In order to motivate the formula consider the trivial case $n = 1$. Then $g = \kappa\, \mathrm{ext}(v) = \kappa\, \mathrm{sgn}\, v\, \mathrm{ext}(|v|) = \kappa(v/|v|)\ \mathrm{ext}(|v|)$, where $\mathrm{ext}(y)$ denotes an arbitrary extremal in $[C(Q)]^*$ for y and $\kappa = \|v\|^{-1}$, clearly yields a norm 1-projection, with v dual to g. It will be shown that for all n,

$$g = v \left[\frac{\kappa\, \mathrm{ext}(|v|)}{|v|}\right] \qquad \left(\text{i.e., } g_i = v_i \left[\frac{\kappa\, \mathrm{ext}(|v|)}{|v|}\right], i = 1, 2,..., n\right), \quad (1)$$

where $\kappa = n\,\||v|\|^{-1}$, $|v| = \sqrt{\Sigma v_i^2}$, and v is dual to g, yields a projection $P_{X,V} = [g]$ with norm $\leqslant \sqrt{n}$.[1] (Note that $P_{X,V}$ is an orthogonal projection.)

[1] Also $g = v\, \mathrm{ext}(|v|)$ by replacing v with $(\sqrt{\kappa/|v|})\, v$ since $|v| = \text{constant}$ on $\mathrm{supp}(\mathrm{ext}(|v|))$.

0021-9045/81/070226-07$02.00/0

EXAMPLE 1. $X = C[a, b]$, $V = [1, x, \ldots, x^{n-1}]$, $n \geqslant 2$. Then $v_i = \sum_{j=0}^{n-1} a_{ij} x^j$ can be chosen so that $g_i = v_i\, d\mu$, $i = 1, \ldots, n$, where $d\mu$ is a measure supported on the n maxima of $|v|^2 = \sum_{i=1}^n v_i^2$. ($|v|^2$ is a non-negative polynomial of degree $2(n-1)$ and therefore has at most n maxima, while $d\mu$ must have at least n points of support.) Thus, $P_{X,V}$ is an interpolating projection.

EXAMPLE 2. $X = C_{2\pi}[-\pi, \pi]$, $V = [1/\sqrt{2}, \sin x, \cos x, \ldots, \sin kx, \cos kx] = [v_1, \ldots, v_n]$, $n = 2k+1$. Then $|v| = \sqrt{(1/2)+k}$, $\operatorname{ext}(|v|) = dx/2\pi$, $\kappa = \sqrt{2+4k}$, $g = v\,(dx/\pi)$, and $P_{X,V}$ is the ordinary Fourier projection.

For $n > 1$ the question of existence (can the duality property be satisfied?) arises as well as the question of norm value. Our method of proof will be to approximate projections from $C(Q)$ to V by special projections $P_p (1 \leqslant p < \infty)$ from $L^p(Q)$ to V having form (1) *relative to* $L^p(Q)$. That is, relative to L^p, $\kappa \operatorname{ext}(|v|) = |v|^{p/q} \in L^q$ $(1/q + 1/p = 1)$ by the Hölder inequality. We define $P_p = [g]$ by

$$g = \frac{v}{|v|} |v|^{p/q} = v|v|^{p-2} \qquad (\text{i.e., } g_i = v_i |v|^{p-2},\ i = 1, \ldots, n), \tag{2}$$

with v dual to g.

Section 2 will show the existence of the projections P_p, as well as a natural iterative Gram–Schmidt-type procedure for obtaining v, suggested by the existence proof. But first, in Section 1, we show $\|P_p\| \leqslant n^{1/2 - 1/p}$, $p \geqslant 2$ wherever P_p exists. Our result then follows by letting $p \to \infty$, and taking $P_{X,V}$ to be a subsequential limit of P_p (restricted to X). In short, P_p allows us to shift our considerations from $C(Q)$ to the simpler space $L^2(Q)$ with a multiplicative loss of at most $\sqrt{n}$ in the norm of $\|P_p\|$. We can therefore write $P_{X,V} = P_\infty|_X$, where $P_\infty = \lim_{p\to\infty} P_p$, whence $\|P_{X,V}\| \leqslant \sqrt{n}$ since $\|P_p\| \leqslant n^{1/2 - 1/p}$, $p \geqslant 2$.

Section 3 will discuss how close the projections P_p, $1 \leqslant p \leqslant \infty$ are to being minimal. It will be emphasized that in some important (regular) cases P_p is minimal. Finally, it will be shown that P_p, $1 \leqslant p \leqslant \infty$, satisfies a certain system of n simple (linear) equations relating $[g]$ to $[v]$, the *variational equations* for minimal projections [1].

As a final preliminary we must note that our procedure above depends on the existence of the space $L^p(Q) = L^p(Q, \nu)$ for some measure ν on Q, consistent with the (weak*–) topology of Q; i.e., for $x \in C(Q)$, $\|x\|_p \to \|x\|_\infty$, as $p \to \infty$. Such a measure can only be guaranteed if X is separable. If X is not separable, Q must be amended to be SV^*, the unit Sphere in V^*, and V is then identified with the set of linear functions on Q. It is easy to check that $\|P_{C(Q),V}\|$ is still an upper bound for the minimal norm among all projections from X onto V, but X is only homomorphically represented in $C(Q)$. Thus the straightforward identification of measures on

Q with X^* which we have in case X is separable becomes more involved in case X is not separable. If X is not separable, once $P_{C(Q),V}$ is determined, the functionals g in V^* corresponding to the measures g on Q must be extended via the Hahn–Banach theorem to appropriate functionals g in X^* in order to obtain $P_{X,V}$. With this understanding we will still write $P_{X,V} = P_\infty|_X$ even if X is not separable. We summarize as follows.

THEOREM. (i) *If X is an arbitrary Banach space and V is an n-dimensional subspace, then the projection $P_{X,V}$ given by formula* (1) *exists* (X *is identified with a subset of* $C(Q)$) *and* $\|P_{X,V}\| \leqslant \sqrt{n}$.

(ii) *The projections P_p, $1 \leqslant p < \infty$, from $L^p(Q)$ onto V given by formula* (2) *exist and* $\|P_p\| \leqslant n^{1/2-1/p}$, $2 \leqslant p < \infty$.

(iii) *The projections P_p, $1 \leqslant p < \infty$ and $P_{X,V} = P_\infty|_X$ satisfy n linear variational equations characterizing a small special family of projections containing the minimal ones.*

Note. (i) follows from (ii) as described above. We therefore demonstrate (ii) and (iii) in the remainder of the paper.

1. THE NORM OF P_p

THEOREM 1. *P_p exists,* $1 \leqslant p < \infty$.

Proof. See Section 2. ∎

THEOREM 2. $\|P_p\| \leqslant n$.

Proof. If P is an arbitrary projection from $L^p(Q)$ onto V, then $\|P\| = \sup_{\|x\|_p=1} \|Px\|_p = \sup_{\|x\|_p=\|y\|_q=1} \int_Q (Px)\, y dv = \sup_{\|x\|_p=\|y\|_q=1} \int_Q \int_Q x(t)[g(t) \cdot v(u)]\, y(u)\, dv(t)\, dv(u)$. Now $|g(t) \cdot v(u)| = |\sum_{i=1}^n g_i(t)\, v_i(u)| \leqslant [\sum g_i^2(t)]^{1/2} [\sum v_i^2(u)]^{1/2} = |g(t)|\, |v(u)|$. Hence $\|P\| \leqslant \||g|\|_q \||v|\|_p$. On the other hand, since v is dual to g, $n = \sum_{i=1}^n \int_Q g_i(t)\, v_i(t)\, dv(t) = \int_Q \sum_{i=1}^n g_i(t)\, v_i(t)\, dv(t) \leqslant \int_Q [\sum g_i^2(t)]^{1/2} [\sum v_i^2(t)]^{1/2}\, dv(t) = \int_Q |g(t)|\, |v(t)|\, dv(t) \leqslant \||g|\|_q \||v|\|_q$. Now let $P = P_p$. The two inequalities in this last chain are then actually equalities since first, for each t, $g_i(t) = cv_i(t)$, where $c = |v(t)|^{p-2}$, $i = 1,\ldots,n$, and second $|g(t)| = [\sum g_i^2(t)]^{1/2} = |v(t)|^{p-2} [\sum v_i^2(t)]^{1/2} = |v(t)|^{p-1} = |v(t)|^{p/q} = \kappa\, \mathrm{ext}(|v(t)|)$. ∎

Remark. Theorem 2 is included because it gives information about P_p ($1 \leqslant p < 2$) not given by Theorem 3 and also because it was the logic of Theorem 2 which made the author first aware of the projection P_p.

THEOREM 3. $\|P_p\| \leqslant n^{1/2-1/p}$, $2 \leqslant p < \infty$.

Proof. P_p is an orthogonal projection with weight $w=|v|^{p-2}$, i.e., $Px=\sum_{i=1}^n \langle x, v_i\rangle_w v_i$ and $\langle v_i, v_j\rangle_w=\delta_{ij}$, where $\langle x,y\rangle_w=\int_Q xywdv$. Letting $\|z\|_{2,w}^2=\langle z,z\rangle_w$, we have

$$\|Px\|_{2,w}^2=\langle Px, Px\rangle_w=\sum_{i=1}^n \langle x, v_i\rangle_w^2 \leqslant \|x\|_{2,w}^2.$$

(Hence, of course, $\|P_p\|=1$ if $p=2$.)

We will next establish the two inequalities

(a) $\|z\|_p \leqslant \|z\|_{2,w}$ for $z \in V$,

and

(b) $\|x\|_{2,w} \leqslant n^{1/2-1/p}\|x\|_p$,

which together with $\|Px\|_{2,w}\leqslant\|x\|_{2,w}$ will finish the proof.

Applying the Hölder inequality with "p" $=p/2$ and "q" $=p/(p-2)$, where $p>2$, we have $\|x\|_{2,w}^2=\int x^2w \leqslant [\int (x^2)^{p/2}]^{2/p}\,[\int w^{p/(p-2)}]^{1-(2/p)}=(\int x^p)^{2/p}(\int |v|^p)^{1-(2/p)}=\|x\|_p^2\cdot n^{1-(2/p)}$, since $\int |v|^p=\int\sum_{i=1}^n v_i^2|v|^{p-2}=\sum_{i=1}^n\langle v_i, v_i\rangle_w=n$, and (b) follows.

For $z\in V$ write $z=\sum_i^n a_iv_i$ whence $\sum_{i=1}^n a_i^2=\langle z,z\rangle_w=\|z\|_{2,w}^2$. Thus $\|z\|_p = \|\sum_{i=1}^n a_iv_i\|_p = [\int |\sum a_iv_i|^2|\sum a_iv_i|^{p-2}]^{1/p} \leqslant [\int |\sum a_iv_i|^2 [(\sum a_i^2)^{1/2}|v|]^{p-2}]^{1/p}=[\sum a_i^2]^{(p-2)/2p}\,[\int(\sum a_iv_i)^2w]^{1/p}=\|z\|_{2,w}^{(p-2)/p}\langle z,z\rangle_w^{1/p}=\|z\|_{2,w}$, and (a) follows. ∎

Note. The author wishes to acknowledge the help of F. T. Metcalf and S. R. Smith in determining the optimal place to apply the Schwarz inequality $|\sum a_iv_i|\leqslant(\sum a_i^2)^{1/2}|v|$ in the chain of inequalities establishing (a) above.

2. The Existence and Construction of P_p

Proof that P_p exists. $P_p=\sum_{i=1}^n g_i(t)\,v_i(u)$, where $g_i(t)=v_i(t)|v(t)|^{p-2}$, $i=1,\ldots,n$, and $\delta_{ij}=\langle g_i, v_j\rangle=\int_Q v_iv_j|v|^{p-2}$ $(1\leqslant i,j\leqslant n)$ is a set of n^2 equations in n^2 unknowns (a_{ij}) if we write $v_i=\sum_{j=1}^n a_{ij}u_j$ $(v=uA)$, where $u=u_1,\ldots,u_n$ is some fixed basis for V. To show that these n^2 equations can be solved for all p, consider the auxiliary function $F[A;p]=\int_Q v_iv_j|v|^{p-2}$ mapping the set of $n\times n$ nonsingular symmetric matrices M_n^s into itself. We want to show that the identity matrix $I=I_n$ is in the image of F for each p. But consider first $p=2$. Then $F[A;2]=I$ can be solved for A by the Gram–Schmidt process. (A can be taken symmetric since a lower triangular matrix L (the usual form for the Gram–Schmidt matrix) becomes symmetric upon changing the basis from u to $u(L^t)^{-1}$.) In fact, therefore, a whole open neighborhood of I in M_n^s is in the range of $F[A;2]=AUA^t$ $(U=(\int_Q u_iu_j))$, namely, the set of all positive definite matrices. Thus, by continuity, for p

sufficiently close to 2, I belongs also to the range of $F[A;p]$. That is, there exists $A = A(p)$ satisfying $F[A;p] = I$ for all p in some neighborhood η of $p = 2$ and range $F[A;2] \supset M_n^s \cap$ [positive definite].

Now let $Z_u = \{q \in Q; u(q) = 0\}$. In the following we can clearly suppose without loss that $\operatorname{int} Z_u$ is empty. But furthermore we can assume that Z_u itself is empty, for otherwise replace Q by $Q - z$, where z is a small open neighborhood of Z_u, apply the theory to $Q - z$, and, in the end, let z approach Z_u, to obtain the conclusion for Q. Thus, under the assumption that Z_u is empty, it is immediate that $F[a;p]$ is infinitely differentiable and in fact analytic[2] in a_{ij} $(1 \leqslant j \leqslant i \leqslant n)$ and p. Hence $A(p)$ is also analytic[2] for p in η by the implicit function theorem. (Since the auxiliary function $f[A;p] = (F[A;p], p)$ is analytic,[2] its inverse is analytic.[2]) But now $A(p)$ can be continued analytically and thus $F[A(p);p] - I \equiv 0$. ∎

Construction of P_p. Consider the map G from V^n to V^n defined as follows. For $v = v_1, \ldots, v_n$ in V^n let $w = |v|^{p-2}$. Then $G(v) = \tilde{v}$ is determined by the Gram–Schmidt (G–S) process with weight w, i.e., $\int_Q \tilde{v}_i \tilde{v}_j \, w dv = \delta_{ij}$ $(1 \leqslant i, j \leqslant n)$. The existence of P_p is equivalent to the existence of a fixed point v of G, suggesting the numerical method of successive approximations to find v:

Numerical procedure. Determine $v^{(1)}$ by the G–S process with weight $w \equiv 1$. Then generate $v^{(k+1)} = Gv^{(k)}$, $k = 1, \ldots$. Each step is a linear (G–S) process.

Comment. The author has carried out many examples in the case $Q = [a, b]$ with the aid of a computer and in all of them, for fixed $p \geqslant 1$, $v^{(k)}$ converged rapidly (to a fixed point v). For $p \geqslant 4$, however, the iterative process had, in general, to be "mollified," i.e., $v^{(k+1)} = \lambda v^{(k)} + (1 - \lambda) Gv^{(k)}$ for some choice of λ.

It should be emphasized that while for simplicity the examples cited are where $Q = [a, b]$, the theory and numerical procedures work the same way for Q of any dimension.

3. The Variational Equations

We emphasize that P_p, $1 \leqslant p \leqslant \infty$, is minimal if $n = 1$ and for all n in the case of the Fourier projection. In this section we show that P_p, $1 \leqslant p \leqslant \infty$, satisfies an important system of n linear equations, the *variational equations* for minimal projections established in [1]. Therefore P_p, if not itself minimal, is at least in a small special family of projections containing those which are

[2]More precisely, has an analytic branch (e.g., if $n = 1$ and $a > 0$ then $F[A;p] = a^p \int |u|^p = 1$ implies $a(p) = (\int |u|^p)^{-1/p}$).

minimal from $L^p(Q)$ ($C(Q)$ if $p=\infty$) onto V. In this sense as well, P_p is close to minimal.

P_p Satisfies $n-1$ "(o)"-Equations

For any orthogonal projection the equations $g_i = wv_i$, $i=1,\ldots,n$ can be rewritten

$$g_i v_{i+1} - g_{i+1} v_i = 0, \qquad i = 1,\ldots, n-1.$$

In [1] it is shown that a minimal projection $P_{\min} = [g]$ satisfies $n-1$ "(o)"-equations:

$$\left[\sum \lambda_{ij}^{(k)} g_i v_j\right]' = 0 \qquad \left(\sum \lambda_{ij}^{(k)} g_i v_j = \text{piecewise constant}\right) \tag{o}$$

for some $n-1$ scalar matrices $n \times n$ matrices $\Lambda^{(k)} = (\lambda_{ij}^{(k)})$, $k=1,\ldots,n-1$.

P_p Satisfies 1 "(*)"-Equation

Assume that V is continuously differentiable on $Q \subset \mathbb{R}^m$. Let $'$ denote any directional derivative. Then $g = wv$, $w = |v|^{p-2} = (v \cdot v)^{(p-2)/2}$ and $g' = w'v + wv'$, where $w' = (p-2)|v|^{p-4} v \cdot v'$. Hence $g' = (p-2)|v|^{p-4}(v \cdot v')\, v + |v|^{p-2} v'$ and so $g' \cdot v = (p-2)|v|^{p-2}(v \cdot v') + |v|^{p-2} v \cdot v' = (p-1)|v|^{p-2} v \cdot v'$ while $g \cdot v' = wv \cdot v' = |v|^{p-2} v \cdot v'$. We conclude that

$$0 = \frac{1}{p} g' \cdot v - \frac{1}{q} g \cdot v' = \sum \left(\frac{1}{p} g_i' v_i - \frac{1}{q} g_i v_i'\right).$$

In [1] it is shown that $P_{\min} = [g]$ satisfies a "(*)"-equation

$$\sum \lambda_{ij} \left(\frac{1}{p} g_i' v_j - \frac{1}{q} g_i v_j'\right) = 0 \tag{*}$$

for some $n \times n$ scalar matrix $\Lambda = (\lambda_{ij})$.

We conclude that P_p satisfies $n-1$ (o)-equations with $\Lambda^{(k)}$ having 1 in the $(k, k+1)$-position, -1 in the $(k+1, k)$-position, 0 elsewhere, $k = 1,\ldots, n-1$, and that P_p satisfies a (*)-equation where $\Lambda = I$, where $'$ denotes an arbitrary directional derivative.

Note (i). The $n-1$ (o)-equations give information about the g_i ($i = 1,\ldots, n$) relative to one another (e.g., $g_i/g_{i+1} = v_i/v_{i+1}$, $i = 1,\ldots, n-1$ (i.e., $g = wv$) if $P_{\min}$ is orthogonal) while the (*)-equations describe the differential structure (i.e., w if $P_{\min}$ is orthogonal) of g relative to v.

Note (ii). Uniqueness questions in the study of minimal projections are closely tied to the dependency of the (*)-equation on the $n-1$ (o)-equations. For example, in the case $n=1$, $p=\infty$, $Q=[a,b]$, the (*)-equation is $gv'=0$ (v' indicates a 2-sided derivative and is thus undefined at endpoints) and therefore $P_{\min}=[g]$ is not determined uniquely if, e.g., $v'=0$ at more than 1 maximum of $|v|$. A second important example is the Fourier projection in the case $p=\infty$, $Q=[-\pi,\pi]$, $v=[1/\sqrt{2}, \sin rx, \cos rx, \sin 2rx, \cos 2rx, \ldots, \sin krx, \cos krx]$. Here the (*)-equation is $g \cdot v'=0$, where $g=wv$. But $g \cdot v' = 0 = w[v \cdot v']$ does not determine w uniquely since $2v \cdot v' = (v \cdot v)' = [\frac{1}{2} + \sum_{j=1}^{k} (\sin^2 jrx + \cos^2 jrx)]' \equiv 0$. Indeed, it is known that for some choices of r and k the Fourier projection though minimal is not unique.

Acknowledgments

The author was introduced to the "minimal projection" problem by Professor Cheney whose help and support, both direct and indirect, is much appreciated. For example, the work [2] has been a great aid and the paper [3] has provided invaluable insight into the L^p-spaces.

Note added in proof. After the above note had been submitted for publication, the author became aware that the projections P_p, $1<p<\infty$, were also developed in a recent paper by D. R. Lewis [6] (where, in fact, it is shown that $\|P_p\| \leqslant n^{|1/2-1/p|}$ by the same argument as in Theorem 3 for $2<p<\infty$ and by a duality argument for $1<p<2$). However, since the existence proof of this note is different and leads to the straightforward construction (iterative Gram–Schmidt process) of the limiting projections P_∞, which are the main focus here, it appeared justified to publish the note without change, as originally written.

References

1. B. L. Chalmers, The variational equations for minimal projections, submitted for publication.
2. E. W. Cheney and K. H. Price, Minimal projections in approximation theory, *in* "Approximation Theory" (A. Talbot, Ed.), pp. 261–289, Academic Press, New York, 1970.
3. C. Franchetti and E. W. Cheney, Minimal projections in $\mathscr{L}_1$-spaces, *Duke Math. J.* **43**(1975), 501–510.
4. F. John, Extremum problems with inequalities as subsidiary conditions, *in* "Studies and Essays, Courant Anniversary Volume" (K. O. Friedrichs, Ed.), pp. 187–204, New York, 1948.
5. M. I. Kadec and M. G. Snobar, Some functionals over a compact Minkowski space, *Math. Notes* **10** (1971), 694–696.
6. D. R. Lewis, Finite dimensional subspaces of L_p, *Studia Math.* **63** (1978), 207–212.